FINAL EXAM

Spring Semester 2021

July 2021

Length of the exam : 3h00 (from 8h15 to 11h15)

Attempt all the questions

First write your name, given names and section :

Name :____________________________ **Given Name :** ____________________________

Section : ____________________

Exercice	Points
1	
2	
3	
4	
5	
Total points :	

1) FOR THIS QUESTION, I DO NOT EXPECT DETAILED IN DEPTH PROOFS BUT SHORT JUSTIFICATION SHOULD BE GIVEN

i)Let $P =$

$$P = \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Is a Markov chain with this transition probability irreducible?

(ii) Consider a Markov chain $(X_n)_{n\geq 0}$ on $\mathbb{Z}_+$ with transition matrix, P, satisfying

$$P_{ij} = 0 \text{ for } |i-j| > 1.$$

Give necessary and sufficient conditions for the chain to be irreducible. Give necessary and sufficient conditions for it to be aperiodic in the case where the chain is irreducible.

(iii) Given a Markov chain $(X_n)_{n\geq 0}$, define a stopping time T.

(iv) True or False? Justify! A Markov chain on $I = \{1,2,3,4,5\}$ must have an invariant distribution.

(v) For transition matrix on $I = \{1,2,3,4\}$

$$P = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix}$$

What is $\mathbb{E}^1(T_4)$ where $T_4 = \inf\{n \geq 0 : X_n = 4\}$.

(vi) Customers arrive at a shop according to a Poisson process of rate 1/hour. Given that 3 customers arrived during time inteval $[2,5]$, what is the probability that exactly one customer arrived in each of the intervals $[2,3]$, $[3,4]$ and $[4,5]$?

(vii) For continuous time Markov chain $(X_t)_{t\geq 0}$ on $\{1,2,3,4\}$, the above matrix is the jump chain and for state i, the expected value of the time spent at i before jumpng is equal to i for $i = 1,2,3,4$. Give the Q-matrix for X.

(viii) Give an example of an irreducible continuous time Markov chain $(X_t)_{t\geq 0}$ which is not positive recurrent but for which the for the jump chain is positive recurrent.

(ix) Consider continuous time Markov chain $(X_t)_{t\geq 0}$ on $\mathbb{N} = \{1, 2, \cdots n, \cdots\}$ with $q_{i\ i+1} = (i+1)^3, \quad q_{i\ 1} = (i+1)^2$ for $i \geq 2$ and all other $q_{i\ j} = 0$ for i, j distinct. Is the chain explosive? Justify.

(i) YES IRREDUCIBLE!

$4 \to 3 \to 2 \to 1.$

$2 \to 4$

$3 \to 4.$

$1 \to 2 \to 4$

(ii) $\forall\, i \geq 0 \begin{pmatrix} P_{i\, i+1} > 0 \\ P_{i+1,\, i} > 0 \end{pmatrix}$ IS A N&S CONDITION FOR IRREDUCIBILITY

AS IF IT FAILS THEN EITHER, FOR SOME i $\quad P^n_{i\, i+1} = 0\ \forall n$ OR $P^n_{i+1,\, i} = 0\ \forall n$

IF IT DOES NOT FAIL $\forall\, i < j$

$p^{j-i}_{ij} \geq P_{i\, i+1} P_{i+1\, i+2} \ldots P_{j-1\, j} > 0$ & SIMILARLY FOR p^{j-i}_{ji}

IF THE CHAIN IS IRREDUCIBLE THEN IT IS APERIODIC $\Leftrightarrow$

$\exists\, i \in \mathbb{Z}_+ : \ P_{ii} > 0$. IF THIS CONDITION FAILS (SO $P_{ii} = 0\ \forall i$) THEN $p^n_{ii} = 0\ \forall\, n$ ODD AS ONE PASSES DETERMINISTICALLY FROM ODD TO EVEN AND FROM EVEN TO ODD

IF $P_{ii} > 0$ FOR SOME $i \Rightarrow i$ HAS PERIOD 1 $\Rightarrow$ BY IRREDUCIBILITY j HAS PERIOD 1 $\forall j$

(iii) GIVEN $(X_n)_{n\geq 0}$ A STOPPING TIME T IS A R.V. (IN EXTENDED SENSE) TAKING VALUES IN $\mathbb{Z}_+ \cup \{\infty\}$

SO THAT $\forall\, n \geq 0,\ n < \infty$ THE EVENT $\{T = n\}$ CAN BE WRITTEN AS $\{(X_0, .. X_n) \in A_n \subseteq I^{n+1}\}$ FOR SOME SUBSET A_n OF I^{n+1}

(iv) Since I is finite so $\exists$ a closed communicating class $J \subseteq I$. On J the transition probabilities give an irreducible M-C on a finite set J.

(v) Let $k_i = \mathbb{E}^i(T_4)$ we want k_1

We have
$$k_2 = 1 + k_1$$
$$k_3 = 1 + \tfrac{1}{2} k_2$$
$$k_1 = 1 + \tfrac{1}{3} k_2 + \tfrac{1}{3} k_3$$
$$\Rightarrow \quad = 1 + \tfrac{1}{3}(k_1 + 1) + \tfrac{1}{3}\left(1 + \tfrac{1}{2}(1 + k_1)\right)$$
$$= \frac{5}{3} + \frac{1}{6} + \frac{k_1}{2}$$
$$\Rightarrow k_1 = 2\left(\tfrac{5}{3} + \tfrac{1}{6}\right) = 11/3$$

(vi) The 3 arrival times are like 3 iid $U[2,5]$ r.v.s

So chance is $3!\, P(U_1 \in [2,3])\, P(U_2 \in [3,4])\, P(U_3 \in [4,5])$
$$= 3!\ \frac{1}{3}\,\frac{1}{3}\,\frac{1}{3} = 2/9$$

(vii)
$$Q = \begin{pmatrix} -q_1 & q_1/3 & q_1/3 & q_1/3 \\ q_2 & -q_2 & 0 & 0 \\ 0 & -\frac{q_3}{2} & -q_3 & -\frac{q_3}{2} \\ \frac{1}{2} q_4 & \frac{1}{4} q_4 & \frac{1}{4} q_4 & -q_4 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1/3 & 1/3 & 1/3 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & +1/6 & -1/3 & +1/6 \\ \frac{1}{8} & \frac{1}{16} & \frac{1}{16} & -1/4 \end{pmatrix} \qquad q_i = \frac{1}{i}\ \forall i\,!$$

(viii) $I = \{0,1,2\ldots\}$

$q_{i\,i+1} = \lambda q_i \quad q_{i\,i-1} = \mu q_i$ FOR $\lambda + \mu = 1 \quad 1 < \frac{\lambda}{\mu} < 2$

$q_i = 2^i$

(ix) THE JUMP CHAIN IS $P_{i\,i+1} = \frac{(i+1)^3}{(i+1)^3 + (i+1)^2} \quad P_{i1} = \frac{(i+1)^2}{(i+1)^3 + (i+1)^2} = \frac{1}{i+2}$

$i \geq 2$

SO $\forall i \quad P^i(T_1 = \infty) = \prod_{k=0}^{\infty}\left(1 - \frac{1}{i+2+k}\right) = 0$

AS $\sum_{k=0}^{\infty} \frac{1}{i+2+k} = \infty$

SO THE CHAIN IS SUCH THAT W.P. 1 IS HIT (AFTER WHICH THE CHAIN STAYS THERE) SO IT IS NOT EXPLOSIVE